Susam's Notes on MIT OCW 18.01

Susam Pal

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These document contains notes taken while studying MIT OpenCourseWare 18.01 Single Variable Calculus, Fall 2006.

Video playlist for this course is available at https://www.youtube.com/playlist?list=PL590CCC2BC5AF3BC1.

The source code for this document is available at https://github.com/spcask/mitocw1801.

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1 Lecture 01: Rate of Change

1.1 Geometric Interpretation

Find the tangent line to $y = f(x)$ at $P = (x_0, y_0)$.

Any line through point (x_0, y_0) has the equation

$$y - y_0 = m(x - x_0).$$

The slope m at (x_0, y_0) can be written as $f'(x_0)$.

Definition 1.1. $f'(x_0)$, the derivative of f at x_0, is the slope of the tangent line to $y = f(x)$ at (x_0, y_0).

Definition 1.2. Tangent line is equal to the limit of secant lines PQ as $Q \to P$ where P is a fixed point.

Let $P = (x_0, f(x_0)$ and $Q = (x_0 + \Delta x, f(x_0 + \Delta x))$.

Then the slope of secant PQ is

$$\frac{\Delta f}{\Delta x}.$$

where $\Delta f = f(x_0 + \Delta x) - f(x_0)$.

The slope of the tangent line at P is

$$m = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

The formula for the derivative of $f(x)$ at P can now be written as

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The formula on the right-hand side is called the *difference quotient.*

Example 1.1. Find the derivative of $f(x) = \frac{1}{x}$ at $x = x_0$.

Solution.

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{1}{x_0 + \Delta x} - \frac{1}{x_0} \right) \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{x_0 - (x_0 + \Delta x)}{(x_0 + \Delta x) x_0} \right) = \frac{-1}{x_0^2}. \end{aligned}$$

Problem 1.1. Find the area of the triangle enclosed by the axes and the tangent to $y = \frac{1}{x}$ at point (x_0, y_0).

Solution. The formula for the tangent is

$$(y - y_0) = f'(x_0)(x - x_0) = \frac{-1}{x_0^2}(x - x_0).$$

The x-intercept of this line can be found by setting $y = 0$ in the above equation as follows:

$$\begin{aligned} 0 - y_0 = \frac{-1}{x_0^2}(x - x_0) \iff \frac{-1}{x_0} &= \frac{-1}{x_0^2}(x - x_0) \\ \implies x_0 &= x - x_0 \\ \iff x &= 2x_0. \end{aligned}$$

By symmetry y-intercept of this line is $y = 2y_0 = \frac{1}{x_0}$.

Therefore, the area of the triangle is

$$\frac{1}{2}(2x_0)(2y_0) = 2x_0y_0 = \frac{2x_0}{x_0} = 2.$$

1.2 More Notations

When we write $y = f(x)$, we also use the notation $\Delta y = \Delta f = f(x + x_0) - f(x)$ and all of the following notations mean the same thing:

$$f' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f = \frac{d}{dx}y.$$

f' is Newton's notation. The others are Leibniz's notation.

Example 1.2. Find the derivative of $f(x) = x^n$ where $n = 1, 2, 3, \ldots$.

Solution.

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left((x + \Delta x)^n - x^n\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(x^n + nx^{n-1}\Delta x + O((\Delta x)^2) - x^n\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(nx^{n-1}\Delta x + O((\Delta x)^2)\right) \\
&= \lim_{\Delta x \to 0} nx^{n-1} + O(\Delta x) \\
&= nx^{n-1}.
\end{aligned}$$

2 Lecture 02: Limits

2.1 Rate of Change

For a function $y = f(x)$, $\frac{\Delta y}{\Delta x}$ represents *average change* in y over the interval Δx and $\frac{dy}{dx}$ represents the *instantaneous rate of change.*

Example 2.1. The variable q represents *charge* and $\frac{dq}{dt}$ represents *current.*

Example 2.2. The variable s represents *distance* and $\frac{ds}{dt}$ represents *speed.*

Problem 2.1 (Pumpkin drop)**.** A pumpkin is dropped from a height of 80 m. Find the average speed of the pumpkin while it falls to the ground and the instantaneous speed of the pumpkin when it hits the ground.

Solution. Let the initial height of the pumpkin be $h_0 = 80m$. Let the height of the pumpkin at time t be h. Then

$$h = h_0 - \frac{1}{2}gt^2 = 80\,\mathrm{m} - (5\,\mathrm{m/s^2}) \cdot t^2.$$

We get $h = 0\,\mathrm{m}$ when $t = 4\,\mathrm{s}$. Therefore, the average speed of the pumpkin is

$$\frac{\Delta h}{\Delta t} = \frac{0 - 80}{4 - 0}\,\mathrm{m/s} = -20\,\mathrm{m/s}.$$

The instantaneous speed of the pumpkin at time t is

$$\frac{d}{dt}h = -gt = (-10\,\mathrm{m/s^2}) \cdot t.$$

The instantaenous speed of the pumpkin when it hits the ground can be found by substituting $t = 4\,\mathrm{s}$ in the above formula:

$$-10\,\mathrm{m/s^2} \cdot 4\,\mathrm{s} = -40\,\mathrm{m/s}.$$

That is $144\,\mathrm{km/h}$.

Example 2.3. The variable T represents *temperature* and $\frac{dT}{dx}$ represents temperature gradient.

Example 2.4 (Sensitivity of measurements)**.** GPS knows the point below the satellite accurately. The GPS device wants to compute its horizontal distance from the point below the satellite. Let us call this distance L. It can measure its distance from the satellite using radio waves. Let us call this distance h. L can be deduced from h.

However, we don't know h exactly. Let the error in h be Δh. The error in L denoted as ΔL is estimated by $\frac{\Delta L}{\Delta h}$ which is approximately $\frac{dL}{dh}$.

2.2 Limits and Continuity

An easy limit is one in which we just need to plug in the limiting value into the formula, for example,

$$\lim_{x \to 4} \frac{x+3}{x^2+1} = \frac{4+3}{4^2+1} = \frac{7}{17}.$$

Derivatives are always harder than this because

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and plugging in $\Delta x = 0$ gives us $\frac{0}{0}$ which is indeterminate.

A left-hand limit is written as $\lim_{x \to x_0^0} f(x)$. It means $\lim_{x \to x_0} f(x)$ for $x < x_0$.

A right-hand limit is written as $\lim_{x \to x_0^+} f(x)$. It means $\lim_{x \to x_0} f(x)$ for $x > x_0$.

Example 2.5. Find the left-hand limit and right-hand limit of

$$f(x) = \begin{cases} x + 1, & x > 0 \\ -x + 2, & x < 0 \end{cases}$$

Solution.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x + 1 = 1,$$
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} -x + 2 = 2.$$

Definition 2.1. f is continuous at x_0 means $\lim_{x \to x_0} f(x) = f(x_0)$, that is, $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$ and $f(x_0)$ is defined.

2.2.1 Jump Discontinuity

A function $f(x)$ is said to have a jump discontinuity at x_0 when $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist but they are not equal.

2.2.2 Removable Discontinuity

A function $f(x)$ is said to have a jump discontinuity at x_0 when $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist and they are equal, but they are not equal to $f(x_0)$.

For example, $g(x) = \frac{\sin x}{x}$ and $h(x) = \frac{1-\cos x}{x}$ have removable discontinuities at $x = 0$.

2.2.3 Infinite Discontinuity

A function $f(x)$ has infinite discontinuity at x_0 if either or both of $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^-} f(x)$ are undefined.

For example, $y = \frac{1}{x}$ has infinite discontinuity at $x = 0$ because $\lim_{x \to 0^-} = -\infty$ and $\lim_{x \to 0^+} = \infty$.

Note that $y = \frac{1}{x}$ is an odd function and its derivative $\frac{dy}{dx} = \frac{-1}{x^2}$ is an even function.

In general, a derivative of an odd function is an even function.

2.2.4 Other (Ugly) Discontinuities

The function $y = \sin \frac{1}{x}$ has an ugly discontinuity at $x = 0$. It oscillates infinitely often as $x \to 0$. There is no left-hand limit or right-hand limit in this case.

2.3 Differentiable Implies Continuous

Theorem 2.1. *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. If f is differentiable at x_0, $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. Therefore,

$$\begin{aligned}\lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\ &= f'(x_0) \cdot 0 = 0.\end{aligned}$$

We have shown that $\lim_{x \to x_0} f(x) - f(x_0) = 0$. This implies that $\lim_{x \to x_0} f(x) = f(x_0)$, that is, $f(x)$ is continuous at x_0. □

3 Lecture 03: Derivatives

3.1 Derivative Formulas

There are two kinds of derivative formulas:

- Specific formulas, for example, $f'(x)$ for $f(x) = x^n, \frac{1}{x}$.
- General formulas, for example, $(u+v)' = u' + v'$, $(cu)' = cu'$.

3.2 Trigonometric Limits

To show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, first draw a unit circle. Draw two lines from the centre of the circle to its circumference. Let the angle between the lines be θ. Then $\sin\theta$ is the length of the perpendicular from the intersection of one line and the circumference to the other line and θ is the length of the arc between the two lines.

Principle: Short pieces of curves are nearly straight.

Therefore, as $\theta \to 0$, the arc length merges with the perpendicular line and

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Using the above result, we can now show that $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} =$

0 as follows:

$$
\begin{aligned}
\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}\frac{1+\cos\theta}{1-\cos\theta} \\
&= \lim_{\theta\to 0} \frac{1-\cos 2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{\sin\theta}{\theta}\frac{\sin\theta}{1+\cos\theta} \\
&= 1\cdot 0 = 0.
\end{aligned}
$$

3.3 Specific Formulas

$$
\begin{aligned}
\frac{d}{dx}\sin x\bigg|_{x=0} &= \lim_{\Delta x\to 0} \frac{\sin(0+\Delta x)-\sin 0}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\sin\Delta x}{\Delta x} = 1.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\cos x\bigg|_{x=0} &= \lim_{\Delta x\to 0} \frac{\cos(0+\Delta x)-\cos 0}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\cos\Delta x - 1}{\Delta x} = 0.
\end{aligned}
$$

Remark: The derivative of $\sin x$ and $\cos x$ at $x = 0$ give all values of $\frac{d}{dx}\sin x$ and $\frac{d}{dx}\cos x$. This can be seen in the next two derivations.

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \sin x \frac{\cos \Delta x - 1}{\Delta x} + \lim_{\Delta x \to 0} \cos x \frac{\sin \Delta x}{\Delta x} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \cos x \frac{\cos \Delta x - 1}{\Delta x} + \lim_{\Delta x \to 0} (-\sin x) \frac{\sin \Delta x}{\Delta x} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}
$$

3.4 Geometric Proof

Draw a unit circle with centre at O. Let N, P, and Q be points on the circumference of the circle arranged in anticlockwise order such that ON is horizontal. Let PQR be a right-angled triangle such that QR is horizontal and PR is vertical.

Let y be the height of P above ON. Let $\Delta y = PR$. Let $\angle NOP = \theta$ and $\angle QOP = \Delta\theta$.

We get arc length $QP = \Delta\theta$, $y = \sin\theta$, and $\angle RPQ = \theta$.

As $\Delta\theta \to 0$, PQ approaches a straight line and $PR = PQ\cos\theta$, so we get

$$\begin{aligned}\frac{d\sin\theta}{d\theta} = \frac{dy}{d\theta} &= \lim_{\Delta\theta\to 0}\frac{\Delta y}{\Delta\theta} \\ &= \lim_{\Delta\theta\to 0}\frac{PR}{PQ} \\ &= \frac{PQ\cos\theta}{PQ} \\ &= \cos\theta.\end{aligned}$$

3.5 General Formulas

3.5.1 Product Rule

$$(uv)' = u'v + uv'.$$

3.5.2 Quotient Rule

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

4 Lecture 04: Chain Rule

4.1 Product Rule

$$(uv)' = u'v + uv'.$$

Proof.

$$\begin{aligned}
(uv)' &= \lim \Delta x \to 0 \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x} \\
&= \lim \Delta x \to 0 \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x} \\
&= \lim \Delta x \to 0 \frac{\begin{array}{c} u(x+\Delta x)v(x+\Delta x) - u(x)v(x) + \\ u(x)v(x+\Delta x) - u(x)v(x+\Delta x) \end{array}}{\Delta x} \\
&= \lim \Delta x \to 0 \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x) + u(x)v(x+\Delta x)}{\Delta x} \\
&= \lim \Delta x \to 0 \frac{\left(u(x+\Delta x) - u(x)\right) v(x+\Delta x) + u(x)\left(v(x+\Delta x) -\right.}{\Delta x}
\end{aligned}$$

□

Example 4.1.

$$\frac{d}{dx}\left(x^n \sin x\right) = nx^{n-1}\sin x + x^n \cos x.$$