Periodic Functions with Sum as Identity Function

Susam Pal

30 Jan 2019

Problem

Find two periodic functions f and g from $\mathbb{R}$ to $\mathbb{R}$ such that their sum $f + g$ is the identity function. The axiom of choice is allowed.

A function f is periodic if there exists $p > 0$ such that $f(x + p) = f(x)$ for all x in the domain.

Solution

The axiom of choice is equivalent to the statement that every vector space has a basis. Since the set of real numbers $\mathbb{R}$ is a vector space over the set of rational numbers $\mathbb{Q}$, there must be a basis $\mathcal{H} \subseteq \mathbb{R}$ such that every real number x can be written uniquely as a finite linear combination of elements of $\mathcal{H}$ with rational coefficients, i.e.,

$$x = \sum_{a \in \mathcal{H}} x_a a$$

where each $x_a \in \mathbb{Q}$ and $\{a \in \mathcal{H} \mid x_a \neq 0\}$ is finite. The set $\mathcal{H}$ is also known as the Hamel basis.

We know that $b_a = 0$ for distinct $a, b \in \mathcal{H}$ because a and b are basis vectors. In the above expansion of x, each x_a is a rational number that appears as the coefficient of the basis vector a. Therefore $(x + y)_a = x_a + y_a$ for all $x, y \in R$. Thus $(x + b)_a = x_a + b_a = x_a + 0 = x_a$. This shows that a function $f(x) = x_a$ is a periodic function with period b for any $b \in \mathcal{H} \setminus \{a\}$.

Let us define two functions:

$$f(x) = \sum_{a \in \mathcal{H} \setminus \{b\}} x_a a, \quad g(x) = x_b b.$$

where $b \in \mathcal{H}$ and $x \in \mathbb{R}$. Let us choose $c \in \mathcal{H}$ such that $c \neq b$. Then $f(x)$ is a periodic function with period b and $g(x)$ is a periodic function with period c. Further,

$$f(x) + g(x) = \left(\sum_{a \in \mathcal{H} \setminus \{b\}} x_a a \right) + x_b b = \sum_{a \in \mathcal{H}} x_a a = x.$$

Thus $f(x)$ and $g(x)$ are two periodic functions such that their sum is the identity function.

References

[1] *A Problem About Periodic Functions*. Project Fermat Mailing List. Jan. 19, 2019. URL: https://groups.google.com/d/msg/projectfermat/WNOVWmCj-gg/lz3ZyyjNFAAJ.

[2] David Radcliffe. *Sum of periodic functions*. Sept. 1, 2013. URL: https://mathblag.wordpress.com/2013/09/01/sums-of-periodic-functions/.

[3] Alex Youcis. *The Dimension of R over Q*. Oct. 22, 2010. URL: https://drexel28.wordpress.com/2010/10/22/the-dimension-of-r-over-q/.